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LETTER TO THE EDITOR

SUSY transformations between diagonalizable and non-diagonalizable Hamiltonians**Boris F Samsonov**

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Online at stacks.iop.org/JPhysA/38/L397**Abstract**

Recently (Samsonov B F and Roy P 2005 *J. Phys. A: Math. Gen.* **38** L249) an explicit example of a \mathcal{PT} -symmetric *non-diagonalizable* Hamiltonian has been given. In this letter we show that such Hamiltonians appear as supersymmetric (SUSY) partners of Hermitian (hence diagonalizable) Hamiltonians and can be transformed back to diagonalizable forms by appropriate SUSY transformations.

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It is well known that there exist non-Hermitian Hamiltonians which cannot be reduced to a diagonal form by changing the basis (so-called *non-diagonalizable Hamiltonians*, see e.g. [1]). To illustrate better our ideas we will consider here only regular Sturm–Liouville problems. The set of eigenfunctions of a non-diagonalizable Hamiltonian is not complete in the corresponding Hilbert space [2, 3]. The *characteristic determinant* has multiple roots. Together with any eigenfunction with a simple eigenvalue coinciding with a multiple root of the characteristic determinant, there exists a set of *associated functions* [2, 3]. The linear hull of eigenfunction and associated functions corresponding to a given value of the energy forms the root subspace (see e.g. [3]). Recently an explicit example of an exactly solvable \mathcal{PT} -symmetric non-diagonalizable Hamiltonian has been given [4].

We have discovered that supersymmetry (SUSY) transformations may convert a Hermitian (hence diagonalizable) Hamiltonian to a non-diagonalizable Hamiltonian, which in particular may possess the \mathcal{PT} symmetry, and vice versa: a possibility which does not appear in the linear algebra. It is related to the possibility of ‘creating’ by SUSY transformations more than one ‘bound state’ at a given non-degenerate value of the energy. Since the energy level is non-degenerate the other state cannot be an eigenfunction of the Hamiltonian but it can be an associated function. In the opposite process, when we ‘delete’ an eigenfunction having a non-zero associated function, the associated function is transformed to a ‘real eigenfunction’. It appears it ‘emerges from background’ and therefore it may be called ‘background eigenfunction’.

We have found that the possibility described above appears if *second order* SUSY transformations or higher are used. It is well known (see e.g. [5]) that there exists a strong

relation between conventional supersymmetric quantum mechanics (SUSY QM, also known in the literature as Witten's SUSY QM), first order Darboux transformations of the Schrödinger equation and Schrödinger's factorization method. We note that higher order extension of Witten's SUSY QM first appeared in [6]. But only after a strong relation between higher order Darboux transformations, factorization of polynomials of one-dimensional Hamiltonians and polynomial supersymmetries of quantum systems was established [7] did the method become an efficient tool in different fields of theoretical and mathematical physics (see recent reviews [8]). In this letter we use the term *SUSY transformation* in its narrow sense as a differential transformation between two exactly solvable Hamiltonians having almost the same (up to a finite number of levels) spectra, and here we briefly review the main properties of these transformations (for details see [7]) we need below. Once such a transformation is found, one can always construct super-Hamiltonian and nilpotent mutually conjugated supercharges. These operators close superalgebra (in general a polynomial superalgebra) [8, 7].

Let us consider two ordinary second order differential equations

$$(h_0 - E)\psi_E(x) = 0 \quad h_0 = -\partial_x^2 + V_0(x) \quad x \in [a, b] \quad (1)$$

$$(h_1 - E)\varphi_E(x) = 0 \quad h_1 = -\partial_x^2 + V_1(x) \quad x \in [a, b] \quad (2)$$

with a and b being finite numbers. We say that the Hamiltonian h_1 is related to h_0 by a second order SUSY transformation if there exists a second order differential operator L intertwining h_0 and h_1 :

$$Lh_0 = h_1L. \quad (3)$$

Once such an operator is found, solutions of equation (2) can be found by applying L to solutions of (1),

$$\varphi_E = L\psi_E \quad \psi_E \notin \ker L.$$

Any second order (in general n th order) intertwining operator may be presented as a superposition of some first order intertwining operators. This means that the second order transformation can always be decomposed into a sequence (or chain) of two first order transformations, but only if this chain is *completely reducible* can one associate a supersymmetric model with every intermediate Hamiltonian of the chain. However, a quadratic supersymmetry can always be associated with h_0 and h_1 if L is a second order differential operator. Second order SUSY transformations we will use below are *irreducible* and therefore there is now way to associate a supersymmetric model with the intermediate Hamiltonian. We will not go into further details about unusual supersymmetric properties related to second order (in general n th order) intertwining operators since this subject is not our main objective and we refer the interested reader to the original [5, 7, 9] and review [8] papers.

Coefficients before the derivative operators in L and the function V_1 should be found from the system of differential equations to which the intertwining relation (3) is reduced. It is remarkable that this system can be completely integrated but the result has a different form depending on the choice of the integration constants. One distinguishes the confluent case, when two integration constants (denoted by α_1 and α_2 below) coincide, from the general case, when they are different. Here we will consider only the latter possibility. Then for the function V_1 one gets

$$V_1 = V_0 - 2[\log W(u_1, u_2)]''. \quad (4)$$

The solution of equation (2) is given by

$$\varphi_E = L\psi_E = W(u_1, u_2, \psi_E)/W(u_1, u_2). \quad (5)$$

Here $W(u_1, u_2)$ and $W(u_1, u_2, \psi_E)$ are 2×2 and 3×3 Wronskians, and u_1, u_2 and ψ_E are solutions to equation (1) corresponding to the eigenvalues α_1, α_2

$$h_0 u_{1,2} = \alpha_{1,2} u_{1,2} \tag{6}$$

and E respectively. α_1 and α_2 are just the integration constants mentioned above and therefore no restrictions are imposed on them except for $\alpha_1 \neq \alpha_2$. Two other integration constants are hidden in the functions u_1 and u_2 since for a given value of α_1 (and α_2) equation (6) has two linearly independent solutions. Expressions (4) and (5) are particular cases of a general result known in the literature as Crum–Krein formulae [10].

Formula (5) defines the operator L for any sufficiently smooth function ψ_E but if ψ_E is a solution to equation (1) other forms of this equation are useful,

$$\varphi_E = (E - \alpha_2)\psi_E + (\alpha_1 - \alpha_2) \frac{W(u_2, \psi_E)}{W(u_1, u_2)} u_1 \tag{7}$$

$$= (E - \alpha_1)\psi_E + (\alpha_1 - \alpha_2) \frac{W(u_1, \psi_E)}{W(u_1, u_2)} u_2. \tag{8}$$

Here the use of equations (6) and (1) has been made to express the second derivatives of the functions u_1, u_2 and ψ_E in terms of the functions themselves. Operator L as given in (7) and (8) maps any two-dimensional space of solutions of equation (1) with the given $E \neq \alpha_1, \alpha_2$ onto the corresponding space of solutions of equation (2). The two-dimensional space $\text{span}\{u_1, u_2\}$ is the kernel of $L, Lu_{1,2} = 0$. Despite this with the help of L one can find solutions of equation (2) corresponding to $E = \alpha_1, \alpha_2$. For this purpose it is necessary to operate with L on the functions

$$\tilde{u}_{1,2} \neq u_{1,2} \quad h_0 \tilde{u}_{1,2} = \alpha_{1,2} \tilde{u}_{1,2}.$$

Using the fact that $W(u_{1,2}, \tilde{u}_{1,2}) = \text{const}$ and putting $\psi_E = \tilde{u}_{1,2}, E = \alpha_{1,2}$ in (7) and (8) one readily gets

$$\varphi_{\alpha_{1,2}} = \frac{u_{2,1}}{W(u_1, u_2)} \quad h_1 \varphi_{\alpha_{1,2}} = \alpha_{1,2} \varphi_{\alpha_{1,2}} \tag{9}$$

where we have omitted an inessential constant factor. It is worth noting that the use of these functions for the next step of transformation gives back the initial Hamiltonian h_0 and hence, the procedure is completely reversible. Our last comment is that (as follows from (4)) for getting non-singular potential differences for $x \in (a, b)$ it is necessary that $W(u_1, u_2) \neq 0$ which will be supposed to be the case.

The properties described above take place irrespective of any boundary value problem related to differential equations (1) or (2). Here using the simplest boundary conditions defining h_0 and h_1 as operators acting in the Hilbert space $L^2(a, b)$ (we will use the same symbols h_0 and h_1 for these operators) we show that a special choice of transformation functions u_1 and u_2 permits us to transform a Hermitian Hamiltonian h_0 into non-Hermitian non-diagonalizable h_1 . Conversely, the inverse transformation converts non-Hermitian and non-diagonalizable h_1 into Hermitian h_0 .

Let us suppose $V_0(x)$ to be a real-valued and sufficiently smooth function for $x \in [a, b]$. Consider two boundary value problems, which we will denote (I) and (II) respectively, defined by equations (1) and (2) and the boundary conditions

$$\psi_E(a) = \psi_E(b) = 0 \tag{10}$$

$$\varphi_E(a) = \varphi_E(b) = 0. \tag{11}$$

It is well known (see e.g. [11]) that the problem (I) has only a discrete, simple and real spectrum of eigenvalues $E = E_n, n = 0, 1, 2, \dots$

We will now formulate conditions for u_1 and u_2 leading to a complex-valued $V_1(x)$ given by (4), and the problem (II) has a real and simple spectrum coinciding with the spectrum of the problem (I) except for one level and if the Hamiltonian h_1 is non-diagonalizable.

It is clear from (7) and (8) that if both u_1 (or equivalently u_2) and $\psi_E, E \neq \alpha_1, \alpha_2$, satisfy the boundary conditions (10) then φ_E satisfies the boundary conditions (11). The only possibility for u_1 to satisfy the zero boundary conditions is to be an eigenfunction of h_0 , $u_1 = \psi_{E_k}$, so that it is (up to an inessential phase factor) real and $\alpha_1 = E_k$, which we shall suppose to be fixed. This means that the Hamiltonian h_1 has the same spectrum as h_0 except maybe for the values $E = \alpha_1, \alpha_2$ but since $u_1 = \psi_{E_k}$ satisfies the boundary conditions (10), the function φ_{α_2} given in (9) is an eigenfunction of h_1 and $E = \alpha_2$ is the spectral point for h_1 . Recalling that we want to keep the real character of the spectrum of h_1 , we have to choose α_2 real also. So, we choose both α_1 and α_2 to be real and the function $u_1 = \psi_{E_k}$ is fixed to be real also but we want to get a complex potential V_1 defined by equation (4). This is possible if u_2 is a complex linear combination of two real linearly independent solutions of equation (1). Let $\alpha_2 (\neq \alpha_1)$ also coincide with a spectral point $E_l (\neq E_k)$ of h_0 ,

$$\alpha_2 = E_l \quad u_2 = \psi_{E_l} + ic\psi_{E_l}^{(2)} \quad c \in \mathbb{R} \quad (12)$$

where ψ_{E_l} satisfies the boundary conditions (10) and $\psi_{E_l}^{(2)}$ is any real solution of equation (1) at $E = E_l$ linearly independent of ψ_{E_l} . We note that $u_2(x) \neq 0 \forall x \in (a, b)$ and for $c \neq 0$ it cannot vanish at both bounds of the interval $[a, b]$.

We claim that if u_1 and u_2 are chosen as described above, the potential V_1 as given in (4) has a spectrum coinciding with the spectrum of V_0 except for the level $E = \alpha_1 = E_k$ which is removed. At the energy $E = \alpha_2 = E_l$ except for an eigenfunction of h_1, φ_{E_l} , there exists an *associated function* χ_{E_l} (see e.g. [2, 3] and also [4]) which we will also call ‘background eigenfunction’, which means that h_1 is non-diagonalizable. It satisfies the inhomogeneous equation

$$(h_1 - E_l)\chi_{E_l} = \varphi_{E_l} \quad \chi_{E_l}(a) = \chi_{E_l}(b) = 0 \quad (13)$$

and also the homogeneous one with the squared Hamiltonian

$$(h_1 - E_l)^2\chi_{E_l} = 0 \quad \chi_{E_l}(a) = \chi_{E_l}(b) = 0 \quad \chi_{E_l} \neq \varphi_{E_l}. \quad (14)$$

We would like to stress that the set $\{\varphi_n\}, n = 0, 1, 2, \dots, n \neq k$ (E_k is ‘removed’), is not complete in $\{L^2(a, b)\}$. To have a complete set one has to add to this set the function χ_{E_l} [2, 3].

As was already pointed out, all spectral points $E_n, n \neq k, l$, of h_0 are also spectral points of h_1 . So, to prove our claim it remains to analyse only the points $E = \alpha_1 = E_k$ and $E = \alpha_2 = E_l$.

One of the solutions $\varphi_{E_k}^{(1)} = \varphi_{\alpha_1}$ of the Schrödinger equation (2) with $E = \alpha_1 = E_k$ is given by (9) from which and the property that the function (12) cannot vanish at both bounds of the interval $[a, b]$, it follows that $\varphi_{E_k}^{(1)}(x)$ does not vanish at both bounds of this interval. A solution of the Schrödinger equation (2) vanishing at one of the bounds, for instance at $x = a$

$$\varphi_{E_k}^{(2)}(x) = \varphi_{E_k}^{(1)}(x) \int_a^x \frac{1}{[\varphi_{E_k}^{(1)}(y)]^2} dy \quad (15)$$

does not vanish at the other bound. This means that $E = \alpha_1 = E_k$ is not a spectral point of h_1 . To get a solution of the Schrödinger equation at $E = \alpha_2 = E_l$ one can use formula (7) with $\psi_E = \psi_{E_l} (\notin \ker L)$ which gives us the function φ_{E_l} satisfying boundary conditions (11),

which means (as was already mentioned) that $E = E_l$ is the spectral point for h_1 . Now we intend to prove that the Hamiltonian h_1 is non-diagonalizable.

Let us consider the solution $\psi_r(E, x)$ of equation (1) fixed by the conditions

$$\psi_r(E, b) = 0 \quad \psi_r'(E, b) = 1.$$

(We denote by prime the derivative with respect to x .) According to (7) the function

$$\varphi_r(E, x) = \frac{1}{E - \alpha_1} L\psi_r(E, x)$$

is such that

$$\varphi_r(E, b) = 0 \quad \varphi_r'(E, b) = 1$$

and

$$\varphi_r(E, a) = \frac{E - \alpha_2}{E - \alpha_1} \psi_r(E, a). \quad (16)$$

Recalling that $\psi_r(E, a)$ as a function of E is an analytic function having only simple zeros at the points of the spectrum of h_0 (see e.g. [11]) and in particular at $E = \alpha_1 = E_k$ and $E = \alpha_2 = E_l$, we conclude that $\varphi_r(E, a)$ is also analytic, $\varphi_r(\alpha_2, a) = 0$ and this zero is double, but $\varphi_r(\alpha_1, a) \neq 0$. The last inequality confirms us once again that $E = \alpha_1$ is not a spectral point for h_1 but the fact the zero $E = \alpha_2$ is double says that h_1 is non-diagonalizable and together with the eigenfunction $\varphi_{E_l}(x) = L\psi_{E_l}(x) \sim \varphi_r(E_l, x)$ there exists an associated function

$$\chi_{E_l} = (\partial\varphi_E/\partial E)_{E=E_l}$$

(see e.g. [2, 3]). It is evident that $\chi_{E_l}(a) = \chi_{E_l}(b) = 0$ and equation (13) it satisfies can be obtained by taking the derivative of equation (2) with respect to E . Since $\varphi_E = L\psi_E$ and L as given in (5) is independent of E , one has

$$\chi_{E_l} = L\tilde{\psi}_{E_l} \quad \tilde{\psi}_{E_l} = (\partial\psi_E/\partial E)_{E=E_l}.$$

The function $\tilde{\psi}_{E_l}$ satisfies the equation

$$(h_0 - E_l)\tilde{\psi}_{E_l} = \psi_{E_l}$$

but it does not satisfy boundary conditions (10) which agrees with the fact that h_0 is a diagonalizable Hamiltonian. Operator L (5) transforms $\tilde{\psi}_{E_l}$ into a solution to equation (13) satisfying the zero boundary conditions, thus transforming it into a ‘background eigenfunction’ of h_1 . We also note that the functions φ_{E_l} and χ_{E_l} form a (non-orthogonal with respect to the usual $L^2(a, b)$ inner product) basis in the two-dimensional root subspace of h_1 corresponding to $E = E_l$ and the function $[\frac{\partial}{\partial E}\varphi_r(E, x)]_{E=E_l}$ belongs to this subspace.

In contrast to the usual SUSY scheme the opposite process, i.e. the ‘deletion’ of the level $E = E_l$ does not actually delete this level. If we take the Hamiltonian h_1 as the initial Hamiltonian for the next second order transformation, leading to the Hamiltonian h_2 , and choose one of the transformation functions defining the transformation operator $L^{(2)}$ of the next step to be equal to φ_{E_l} , actual eigenfunction at $E = E_l$ is deleted but the associated function χ_{E_l} ‘comes out of the background’ and becomes a true eigenfunction of h_2 at $E = E_l$. This statement is readily verified if one acts by $L^{(2)}$, which is constructed in a similar way as $L = L^{(1)}$ given in (5) and intertwines now h_1 and h_2 , on both sides of (13), takes into account the intertwining relation $L^{(2)}h_1 = h_2L^{(2)}$ and the property $L^{(2)}\varphi_{E_l} = 0$. If the non-diagonalizable Hamiltonian h_1 has only one associated function, it is transformed in this way into a diagonalizable h_2 .

The simplest example illustrating the possibilities described above is the boundary value problem with the zero initial potential $V_0(x) = 0$. We will choose $a = -\pi$ and $b = \pi$. The solutions of the boundary value problem (I) are well known, for instance, its discrete spectrum is $E = E_n = \frac{1}{4}n^2, n = 1, 2, \dots$

Let us choose

$$u_1 = \sin(Ax) \quad u_2 = \exp(-iBx) \quad A, B \in \mathbb{R}.$$

Formula (4) gives us the following \mathcal{PT} -symmetric Hamiltonian:

$$V_1 = \frac{2A^2(A^2 - B^2)}{[\cos(Ax) - iB \sin(Ax)]^2}. \quad (17)$$

For $A = 1$ the function u_1 coincides with the first excited state of h_0 and for $B \neq n/2$ this potential is diagonalizable with the spectrum $E = E_n = \frac{1}{4}n^2, n = 1, 3, 4, 5, \dots$ and $E_{\alpha_2} = B^2$. For $B = 2$ the function u_2 is a complex linear combination of the fourth excited state and another solution of equation (1) with $V_0(x) = 0$ at the same energy and the level E_{α_2} merges with the existing level $E = 4$ which 'goes to background'. The potential (17) becomes non-diagonalizable with the discrete spectrum $E = E_n = \frac{1}{4}n^2, n = 1, 3, 4, 5, \dots$, studied in detail in [4].

Now we would like to illustrate the possibility of transforming the non-diagonalizable potential (17) at $A = 1$ and $B = 2$ into a diagonalizable one. We choose V_1 as the initial potential and take $u_1 = \varphi_4$ and $u_2 = \varphi_{\text{left}}$ where φ_{left} is such that $\varphi_{\text{left}}(-\pi) = 0$. This yields the following potential:

$$V_2 = \frac{(\kappa^2 - 1)[\kappa^2 - 1 - \kappa^2 \cos(2x) + \cos(2\kappa x + 2\kappa\pi)]}{[\kappa \cos(\kappa x + \kappa\pi) \sin x - \sin(\kappa x + \kappa\pi) \cos x]^2} \quad \kappa \neq 1 \quad (18)$$

where we denoted $\alpha_2 = \kappa^2$. It is regular $\forall x \in (-\pi, \pi)$ provided $0.5 \leq \kappa \leq 1.5, \kappa \neq 1$ and has the spectrum $E = E_n = \frac{n^2}{4}, n = 1, 3, 4, \dots$ and $E = \alpha_2 = \kappa^2$. For $\kappa = 1$, $\varphi_{\text{left}} = L\psi_{\text{left}} = 0$ and to realize the transformation with $\alpha_2 = 1$ one has to use solutions obtained with the help of formula (9) as transformation functions. This corresponds to the backward transformation from V_1 to $V_0 = 0$ and hence one gets $V_2 = 0$. For real κ the potential (18) is real and corresponds to the Hermitian (hence diagonalizable) Hamiltonian $h_2 = -\partial_x^2 + V_2$. So, we have transformed the non-diagonalizable Hamiltonian h_1 to the diagonalizable h_2 . One can also transform h_1 into a non-Hermitian diagonalizable h_2 by choosing a complex linear combination of two linearly independent solutions of equation (2) corresponding to the same value of $E = \alpha_2$ as transformation function u_2 .

Our last example is related to the possibility of enlarging the root subspace corresponding to $E = 4$ of the potential (17) at $A = 1$ and $B = 2$ from dimension 2 to dimension 3. To this end we take

$$u_1 = \varphi_{E_1} \quad u_2 = \frac{9 - e^{-2ix}}{1 - 3e^{2ix}}$$

which yields the potential

$$V_2 = 6 \frac{25 e^{ix} + 324 e^{2ix} + 1350 e^{3ix} + 2500 e^{4ix} + 2025 e^{5ix}}{(3 + 25 e^{ix} + 81 e^{2ix} + 75 e^{3ix})^2}.$$

It has the spectrum $E = \frac{n^2}{4}, n = 3, 4, 5, \dots$

We hope that the possibility of transforming non-diagonalizable \mathcal{PT} -symmetric Hamiltonians to diagonalizable forms may find application in complex quantum mechanics which is currently under development.

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